## Differentiation Review

Fall 2022

The slope of the secant line for a function through the points $\left(x_{1}, f\left(x_{1}\right)\right)$ and $\left(x_{2}, f\left(x_{2}\right)\right)=\left(x_{1}+h, f\left(x_{1}+h\right)\right)$ is

$$
m=\frac{f\left(x_{2}\right)-f\left(x_{1}\right)}{x_{2}-x_{1}}=\frac{f(x+h)-f(x)}{h} .
$$

The slope of the tangent line for a function at the point $(x, f(x))$ is given by

$$
f^{\prime}(x)=\lim _{h \rightarrow 0} \frac{f(x+h)-f(x)}{h} .
$$

When this limit exists, we say that the function is differentiable and we call this limit the derivative of the function $f$ at $x$.

## Polynomials and Powers

We can apply the rules of differentiation to arrive at some general rule for finding derivatives of polynomials, or any function involving $x$ raised to a particular power. Let is begin with a simple function: $f(x)=x^{3}$. Applying the limit definition yields:

$$
\begin{aligned}
f^{\prime}(x) & =\lim _{h \rightarrow 0} \frac{(x+h)^{3}-x^{3}}{h} \\
& =\lim _{h \rightarrow 0} \frac{x^{3}+3 x^{2} h+3 x h^{2}+h^{3}-x^{3}}{h} \\
& =\lim _{h \rightarrow 0} \frac{3 x^{2} h+3 x h^{2}+h^{3}}{h} \\
& =\lim _{h \rightarrow 0} 3 x^{2}+3 x h+h^{2} \\
& =3 x^{2} .
\end{aligned}
$$

Observe that the original exponent, 3, moved towards the front and that the power of x decremented by one to that of 2 . In fact, this pattern holds for any real number, even fractions and irrational numbers. The law is as follows for some real number $b \neq 0$,

$$
\frac{d}{d x}\left(x^{b}\right)=b x^{b-1}
$$

Combining this with the sum rule for differentiation, we can now establish the general rule for polynomials. Observe: If

$$
p(x)=a_{n} x^{n}+a_{n-1} x^{n-1}+\cdots+a_{2} x^{2}+a_{1} x+a_{0}
$$

then

$$
p^{\prime}(x)=n a_{n} x^{n-1}+(n-1) a_{n-1} x^{n-2}+\cdots+2 a_{2} x+a_{1} .
$$

This holds whether $n$ is an integer, a fraction, or an irrational number.

## Exponential Functions

By far the most important function in mathematics is the exponential: $f(x)=e^{x}$. Not only is this function infinitely differentiable, but its derivative is itself. That is

$$
\frac{d}{d x}\left(e^{x}\right)=e^{x}
$$

For exponential functions of other bases, the rule is

$$
\frac{d}{d x}\left(a^{x}\right)=a^{x} \ln a .
$$

## Logarithmic Functions

Recall that the function $f(x)=\ln (x)$ is merely the inverse of $g(x)=e^{x}$. It is possible to derive differentiation rules from this relationship, however, we will save that for later. Just acknowledge that

$$
\frac{d}{d x}(\ln x)=\frac{1}{x}
$$

For logarithmic functions of other bases, we have

$$
\frac{d}{d x}\left(\log _{a} x\right)=\frac{1}{x \ln a}
$$

## Trigonometric Functions

The derivatives of each of the six trigonometric functions are
$\frac{d}{d x}(\sin x)=\cos x$

$$
\frac{d}{d x}(\cos x)=-\sin x
$$

$$
\frac{d}{d x}(\tan x)=\sec ^{2} x
$$

$$
\begin{aligned}
\frac{d}{d x}(\csc x) & =-\csc x \cot x \\
\frac{d}{d x}(\sec x) & =\sec x \tan x \\
\frac{d}{d x}(\cot x) & =-\csc ^{2} x .
\end{aligned}
$$

## Hyperbolic Functions

A special class are called hyperbolic functions. Recall that the parametric equations $(\cos (t), \sin (t))$ trace a unit circle on a graph. Likewise, the parametric equations $(\cosh (t), \sinh (t))$ trace a unit hyperbola. Equivalently, this unit hyperbola is traced by the relation $x^{2}-y^{2}=1$, much in the same way that the relation $x^{2}+y^{2}=1$ traces a unit circle. With this informal background, let us define the two most important hyperbolic functions:
$\cosh x=\frac{e^{x}+e^{-x}}{2}$

$$
\sinh x=\frac{e^{x}-e^{-x}}{2}
$$

Using this definition, one can show
$\frac{d}{d x} \cosh (x)=\sinh (x)$

$$
\frac{d}{d x} \sinh (x)=\cosh (x)
$$

We extend the definition of these hyperbolic functions to include other hyperbolic functions. Observe:

$$
\tanh x=\frac{\sinh x}{\cosh x} \quad \operatorname{csch} x=\frac{1}{\sinh x} \quad \operatorname{sech} x=\frac{1}{\cosh x} \quad \operatorname{coth} x=\frac{1}{\tanh x}
$$

Equipped with these new definitions, we can deduce that
$\frac{d}{d x}(\tanh x)=\operatorname{sech}^{2} x$
$\frac{d}{d x}(\operatorname{sech} x)=-\operatorname{sech} x \tanh x$
$\frac{d}{d x}(\operatorname{csch} x)=-\operatorname{csch} x \operatorname{coth} x$

$$
\frac{d}{d x}(\operatorname{coth} x)=-\operatorname{csch}^{2} x
$$

## Product Rule

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is not the product of their derivatives.

If $u$ and $v$ are differentiable at $x$, then so is their product $u v$, and

$$
\frac{d}{d x}(u v)=u v^{\prime}+u^{\prime} v
$$

## Quotient Rule

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two differentiable functions is not the quotient of their derivatives.

If $u$ and $v$ are differentiable at $x$ and $v(x) \neq 0$, then the quotient $\frac{u}{v}$ is differentiable at $x$, and

$$
\frac{d}{d x}\left(\frac{u}{v}\right)=\frac{v u^{\prime}-u v^{\prime}}{v^{2}}
$$

## Chain Rule

If $f(u)$ is differentiable at the point $u=g(x)$ and $g(x)$ is differentiable at $x$, then the composite function $(f \circ g)(x)=f(g(x))$ is differentiable at $x$, and

$$
(f(g(x)))^{\prime}=f^{\prime}(g(x)) g^{\prime}(x)
$$

In Leibniz notation, if $y=f(u)$ and $u=g(x)$, then

$$
\frac{d y}{d x}=\frac{d y}{d u} \cdot \frac{d u}{d x}
$$

where $\frac{d y}{d u}$ is evaluated at $u=g(x)$.

## Implicit Differentiation

The chain rule will now play a very important role. Recall that if we have a function within a function, the derivative formula reads

$$
\frac{d}{d x}(f(g(x)))=f^{\prime}(g(x)) g^{\prime}(x) .
$$

Notice here that $g$ depends on $x$, and is in turn within the structure of $f$. Using this notion, we can apply the chain rule to situations when no such function is given explicitly. We are left to deal with some sort of implicit relation, like the following: $x^{2}-\cos (y)-y+x^{3}=27$. Notice that $y$ is not given in terms of $x$. Not only that, any attempt to isolate $y$ may prove to be too difficult. How do we find $y^{\prime}$ ? The answer lies in the power of the chain rule. Recall that $y$ depends on $x$, now carefully observe the following computations:

$$
\begin{gathered}
x^{2}-\cos (y)-y+x^{3}=27 \\
\Rightarrow 2 x+\sin (y) y^{\prime}-y^{\prime}+3 x^{2}=0 .
\end{gathered}
$$

We stop here to take note of the second term. Notice that first, we differentiated the cosine. Then, since $y$ depends on $x$, we need the derivative of $y$ with respect to $x$ in order to bring the chain rule to its completion. The behavior of any implicit relation between $y$ and $x$ is similar. Now let us finish the problem by solving for $y^{\prime}$ :

$$
\begin{gathered}
2 x+\sin (y) y^{\prime}-y^{\prime}+3 x^{2}=0 \\
\Rightarrow y^{\prime}=-\frac{2 x+3 x^{2}}{\sin y-1} .
\end{gathered}
$$

## Examples

Find the first derivative for each of the following functions.

1. $f(x)=\tanh x^{2}$
2. $g(x)=e^{-x^{2}-3 x+1}$
3. $h(x)=\ln x \sin x$
4. $F(x)=\ln \sqrt{x}$
5. $f(x)=x e^{3 x}$
6. $y=x^{-\frac{3}{2}}$
7. $h(x)=\sec \left(x^{2}+2 x\right)$
8. $f(x)=e^{-2 x^{3 / 2}}$
9. $G(x)=x^{1 / 3}-x^{-2}$
10. $H(t)=\frac{1}{2}\left(e^{t}+e^{-t}\right)$
11. $f(x)=\left(x^{2}+2\right)(x-3)^{-2}$
12. $y=e^{2 x} \cos 3 x$
13. $g(x)=\sin ^{3} x$
14. $f(x)=\frac{\sin x}{x^{2}+1}$
15. $g(t)=\frac{t+1}{t^{2}-2 t+1}$

Find the first three derivatives of the following functions
16. $f(x)=\cos e^{x}$

Find $\frac{d y}{d x}$ implicitly.
18. $3 x^{2} y^{2}+\sin y=3$
19. $x e^{y}+\tan x=0$
20. $e^{y}(\sin y-\cos y)=-e^{x}$
17. $f(x)=x^{2}+x \sin x-\cos x$
21. $\ln y-x^{2} y+y^{2} x=0$

## Key

1. $\frac{d}{d x} \tanh x^{2}=2 x \operatorname{sech}^{2} x^{2}$
2. $\frac{d}{d x} e^{-x^{2}-3 x+1}=(-2 x-3) e^{-x^{2}-3 x+1}$
3. $\frac{d}{d x} \ln x \sin x=\frac{1}{x} \sin x+\ln x \cos x$
4. $\frac{d}{d x} \ln \sqrt{x}=\frac{1}{\sqrt{x}} * \frac{1}{2 \sqrt{x}}=\frac{1}{2 x}$
5. $\frac{d}{d x} x e^{3 x}=e^{3 x}+3 x e^{3 x}$
6. $\frac{d}{d x} x^{-\frac{3}{2}}=-\frac{3}{2} x^{\frac{-5}{2}}$
7. $\frac{d}{d x} \sec \left(x^{2}+2 x\right)=\left(\sec \left(x^{2}+2 x\right) \tan \left(x^{2}+2 x\right)\right)(2 x+2)$
8. $\frac{d}{d x} e^{-2 x^{3 / 2}}=e^{-2 x^{3 / 2}}\left(-3 x^{1 / 2}\right)$
9. $\frac{d}{d x} x^{1 / 3}-x^{-2}=\left(\frac{1}{3} x^{\frac{-2}{3}}\right)+2 x^{-3}$
10. $\frac{d}{d t} \frac{1}{2}\left(e^{t}+e^{-t}\right)=\frac{1}{2}\left(e^{t}-e^{-t}\right)=\sinh t$
11. $\frac{d}{d x}\left(x^{2}+2\right)(x-3)^{-2}=2 x(x-3)^{-2}+\left(x^{2}+2\right)(-2)(x-3)^{-3}$
12. $\frac{d}{d x} e^{2 x} \cos 3 x=2 e^{2 x} \cos 3 x-3 e^{2 x} \sin 3 x$
13. $\frac{d}{d x} \sin ^{3} x=3 \sin ^{2} x \cos x$
14. $\frac{d}{d x} \frac{\sin x}{x^{2}+1}=\frac{\cos x\left(x^{2}+1\right)-2 x \sin x}{\left(x^{2}+1\right)^{2}}$
15. $\frac{d}{d t} \frac{t+1}{t^{2}-2 t+1}=\frac{\left(t^{2}-2 t+1\right)-(t+1)(2 t-2)}{\left(t^{2}-2 t+1\right)^{2}}$

Find the first three derivatives of the following functions
16. $\frac{d}{d x} \cos e^{x}=-e^{x} \sin e^{x}$

$$
\begin{aligned}
& \frac{d^{2}}{d x^{2}} \cos e^{x}=-e^{x} \sin e^{x}-e^{2 x} \cos e^{x} \\
& \frac{d^{3}}{d x^{3}} \cos e^{x}=-e^{x} \sin e^{x}-e^{2 x} \cos e^{x}-2 e^{2 x} \cos e^{x}+e^{3 x} \sin e^{x}
\end{aligned}
$$

17. $\frac{d}{d x} x^{2}+x \sin x-\cos x=2 x+\sin x+x \cos x+\sin x$ $\frac{d^{2}}{d x^{2}} x^{2}+x \sin x-\cos x=2+2 \cos x+\cos x-x \sin x$ $\frac{d^{3}}{d x^{3}} x^{2}+x \sin x-\cos x=0-3 \sin x-\sin x-x \cos x$

Find $\frac{d y}{d x}$ implicitly.
18. $\frac{d}{d x} 3 x^{2} y^{2}+\sin y=3$
$6 x y^{2}+6 x^{2} y \frac{d y}{d x}+\cos y \frac{d y}{d x}=0$
$\frac{d y}{d x}=\frac{-6 x y^{2}}{6 x^{2} y+\cos y}$
19. $\frac{d}{d x} x e^{y}+\tan x=0$
$e^{y}+x e^{y} \frac{d y}{d x}+\sec ^{2} x=0$
$\frac{d y}{d x}=\frac{-\sec ^{2} x-e^{y}}{x e^{y}}$
20. $\frac{d}{d x} e^{y}(\sin y-\cos y)=-e^{x}$
$\frac{d y}{d x} e^{y}(\sin y-\cos y)+e^{y}\left(\frac{d y}{d x} \cos y+\frac{d y}{d x} \sin y\right)=-e^{x}$
$\frac{d y}{d x}=\frac{-e^{x}}{2 e^{y} \sin y}$
21. $\frac{d}{d x} \ln y-x^{2} y+y^{2} x=0$
$\frac{d y}{d x} \frac{1}{y}-2 x y-x^{2} \frac{d y}{d x}+y^{2}+2 x y \frac{d y}{d x}=0$
$\frac{d y}{d x}=\frac{2 x y-y^{2}}{\frac{1}{y}+2 x y-x^{2}}$

