A Quick Review of Integration.

1. Integrating by sight: recognizing the integrand as the derivative of something else.
2. Checking your work: a. by taking the derivative and or b. with Mathematica.
3. Indefinite integrals (general antiderivatives) and definite integrals (evaluate the value of an antiderivative at the lower bound and subtract from the value of that antiderivative at the upper bound.)
e.g $\int x\,dx = \frac{1}{2}x^2 + C$ so $\int_3^7 x\,dx = \frac{1}{2}x^2|_3^7 = \frac{1}{2}7^2 - \frac{1}{2}3^2 = \frac{49}{2} - \frac{9}{2} = \frac{40}{2} = 20$
4. Constant multiple rule $\int kf(x)dx = k\int f(x)dx$ and sum-difference rule $\int(\mathrm{f(x)} + \mathrm{g(x)})\mathrm{dx} = \int \mathrm{f(x)dx} + \int \mathrm{g(x)dx}$
5. *u*-substitution: the chain rule backwards $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ (the derivative of the outside evaluated at the inside times the derivative of the inside.) To go backwards, look for one function, $g(x)$, inside another function, $f(x)$, with $g'(x)$ multiplying dx. Set $u = g(x)$. Then $du = g'(x)dx$.
6. Integration by parts: the product rule in reverse. The product rule says $\frac{d}{dx}(\mathrm{f}(x)\mathrm{g}(x) = f'(x)\mathrm{g}(x) + \mathrm{f}(x)g'(x)$
If you integrate both sides w.r.t. x, you get $f(x)g(x) = \int f'(x)g(x)dx + \int (f(x)g'(x)dx$. Rearranging gives $\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$. The lets you trade in one integral for a product minus a different integral. If the new integral, is easier, you win! When integrating a product, look for one factor that gets nicer (or no worse) when you differentiate it and one factor that gets nicer (or no worse) when you integrate it.
7. Advanced techniques: using a trig identity or trig substitution, partial fraction decomposition.

1. Sight Practice: write the derivatives of $x^n, e^{\mathrm{kx}}, ln|x|, sin(x), cos(x), tan(x), sec(x), cot(x)$, and csc(x), then use these to find $\int x^n dx$, $\mathrm{n} \neq -1$,
$\int x^{-1}\,\mathrm{dx}$, $\int e^{kx}\,\mathrm{dx}$, $\int sin(x)\,dx$, $\int cos(x)\,dx$,

$\int sec^2(x)\,dx$, $\int sec(x)\,tan(x)\,dx$, $\int csc^2(x)\,dx$, $\int csc(x)\,cot(x)\mathrm{dx}$

2a. Check the following antiderivatives by differentiating the right-hand side:
$\int \frac{x+1}{x^2+2x}dx = \frac{1}{2}ln|x^2 + 2x| + C$ $\int tan(x)\,dx = -ln|cos(x)| + \mathrm{C}$ $\int cot\,(x)\,dx = ln\,|\,sin\,(x)| + C$

2b. Install Mathematica using the instructions here https://www.byui.edu/mathematics/student-resources/mathematica
Try typing Integrate[Cot[x], x] and then Shift-Enter. Notice that Mathematica always capitalizes names of built-in functions (like Integrate and Cot.) Arguments to functions go in square brackets "[" and "]", not parentheses. What's the difference between executing Integrate[y*Cot[x], x] and executing Integrate[y*Cot[x], y]? Why?
Now type D[x^5, x] and then Shift-Enter. What does the built-in function "D" do?

3. Evaluate the following definite integrals: $\int_0^\pi sin(x)\,dx$, $\int_1^e \frac{1}{x}dx$, $\int_0^b e^{-3x}\,dx$, $\int_0^\infty e^{-3x}\,dx$

4. Evaluate $\int_0^5 2x^3 - 5x + 4dx$
5. Evaluate the following using a *u* substitution

$\int x\sin(2x^2)dx$ $\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}dt$ $\int x^3(x^4-1)^2\,dx$ $\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$

$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}dx$ $\int \tan^2 x \sec^2 x\,dx$ $\int \frac{6\cos t}{(2+\sin t)^3}dt$ $\int \frac{\sec z \tan z}{\sqrt{\sec z}}dz$

6. Evaluate the following using integration by parts

$\int t\cos(\pi t)dt$ $\int x^2 \sin x\,dx$ $\int_1^e x^3 \ln x\,dx$ $\int \sin^{-1} y\,dy$ $\int e^{-y}\cos y\,dy$

KEY:

1. Sight Practice: write the derivatives of $\frac{d}{dx}x^n = nx^{n-1}$,

$\frac{d}{dx}e^{kx} = ke^{kx}$, $\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$,

$\frac{d}{dx}\tan(x) = \sec^2(x)$, $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, $\frac{d}{dx}\cot(x) = -\csc^2(x)$, $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$

then use these to find $\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$, $\int x^{-1}\, dx = \ln|x| + C$,

$\int e^{kx}\, dx = \frac{e^{kx}}{k} + C$, $\int \sin(x)\, dx = -\cos(x) + C$, $\int \cos(x)\, dx = \sin(x) + C$, $\int \sec^2(x)\, dx = \tan(x) + C$,

$\int \sec(x)\tan(x)\, dx = \sec(x) + C$, $\int \csc^2(x)\, dx = -\cot(x) + C$, $\int \csc(x)\cot(x)\, dx = -\csc(x) + C$

2a. Check the following antiderivatives by differentiating the right-hand side:

$\int \frac{x+1}{x^2+2x} dx = \frac{1}{2}\ln|x^2 + 2x| + C$ $\int \tan(x)\, dx = -\ln(\cos(x)) + C$ $\int \cot(x)\, dx = \ln|\sin(x)| + C$

2b. Install Mathematica using the instructions here https://www.byui.edu/mathematics/student-resources/mathematica
Try typing Integrate[Cot[x], x] and then Shift-Enter. Notice that Mathematica always capitalizes names of built-in functions (like Integrate and Cot.) Arguments to functions go in square brackets "[" and "]", not parentheses. What's the difference between executing Integrate[y*Cot[x], x] and executing Integrate[y*Cot[x], y]? Why?
Now type D[x^5, x] and then Shift-Enter. What does the built-in function "D" do?

3. Evaluate the following definite integrals: $\int_0^{\pi} \sin(x)\, dx = -\cos(x)|_0^{\pi} = -(-1) - (-1) = 2$,

$$\int_1^e \frac{1}{x} dx = \ln|x|\,|_1^e = \ln(e) - \ln(1) = 1 - 0 = 1,$$

$$\int_0^b e^{-3x}\, dx = \left.\frac{-e^{-3x}}{3}\right|_0^b = \frac{e^{-3b}}{3} - \frac{-e^0}{3} = \frac{1 - e^{-3b}}{3},$$

A definite integral is only defined over a finite interval, but when one of the bounds is infinite, we can integrate up to a finite bound, and then take the limit as that bound goes to infinity: $\int_0^{\infty} e^{3x}\, dx = \lim_{b\to\infty} \int_0^b e^{3x}\, dx = \lim_{b\to\infty} \frac{1-e^{-3b}}{3} = \frac{1}{3}$

4. Evaluate $\int_0^5 2x^3 - 5x + 4dx = 2\int_0^5 x^3 - \int_0^5 5x + 4\int_0^5 dx\ = 2\frac{x^4}{4}\Big|_0^5 - 5\frac{x^2}{2}\Big|_0^5 + 4x|_0^5 = \frac{625}{2} - \frac{125}{2} + 20 = 270$
Check: Integrate$[2 * x\text{^}3 - 5x + 4, \{x, 0, 5\}]$

5. Evaluate the following using a u substitution

$\int x\sin(2x^2)dx$ Notice the derivative of $2x^2$ is $4x$ which is close to the xdx we have. So set $u = 2x^2$

Then $du\ =\ 4xdx$ or $xdx = \frac{1}{4}du$

Now rewrite the integral in terms of u to get $\int \frac{1}{4}sin(u)\,du = \frac{-cos(u)}{4} + C$. Substituting back gives $-\frac{cos(2x^2)}{4} + C$

$\int \left(1-\cos\frac{t}{2}\right)^2 \sin\frac{t}{2}dt$ Because the derivative of cosine is -sine, and we have sine multiplying dt, we set $u = 1 - cos\left(\frac{t}{2}\right)$. Then du $= \frac{1}{2}sin\left(\frac{t}{2}\right)dt$ or $sin\left(\frac{t}{2}\right)dt = 2du$. Now use our expressions for u and du to write the integral completely in terms of u: $\int 2u^2du = \frac{2u^3}{3} + C$. Substituting back gives $\frac{2\left(1-cos\left(\frac{1}{2}t\right)\right)^3}{3} + C$

$\int x^3(x^4-1)^2\,dx$. The derivative of x^4 is $4x^3$, which is close to x^3, so set $u = x^4 - 1$. Then $du = 4x^3dx$ or $x^3dx = \frac{1}{4}du$. Substituting to write the integral entirely in terms of u gives $\int \frac{u^2}{4}du = \frac{u^3}{12} + C = \frac{(x^4-1)^3}{12} + C$

$\int \frac{1}{x^2}\cos^2\left(\frac{1}{x}\right)dx$. The derivative of $\frac{1}{x}$ is $\frac{-1}{x^2}$ so set $u = \frac{1}{x}$. Then $du = \frac{-1}{x^2}dx$ or $\frac{1}{x^2}dx = -du$. Using this to write the integral completely in terms of u gives $-\int cos^2(u)\,du$. Integrating $cos^2(u)$ requires the trig identity $cos^2(u) = \frac{1}{2} + \frac{1}{2}cos(2u)$ so $-\int cos^2(u)du = -\frac{1}{2}\int du - \frac{1}{2}\int cos(2u)\,du = -\frac{1}{2}u - \frac{1}{4}sin(2u) + C = -\frac{1}{2x} - \frac{1}{4}sin\left(\frac{2}{x}\right) + C$

$\int \frac{(1+\sqrt{x})^3}{\sqrt{x}}dx$ The derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ so set $u = 1 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$ or $\frac{1}{\sqrt{x}}dx = 2du$. Writing the integral in terms of u gives $\int 2u^3du = \frac{1}{2}u^4 + C = \frac{(1+\sqrt{x})^4}{2} + C$

$\int \tan^2 x\sec^2 x\,dx$ The derivative of $tan(x)$ is $sec^2(x)$ so set $u = tan(x)$. Then $du = sec^2(x)\,dx$. Writing the integral in terms of just u gives $\int u^2du = \frac{u^3}{3} + C = \frac{tan^3(x)}{3} + C$

$\int \frac{6\cos t}{(2+\sin t)^3}dt$. The derivative or $2 + sin(t)$ is $cos(t)$,so set $u = 2 + sin(t)$. Then $du = cos(t)\,dt$ and the integral becomes $\int \frac{6}{u^3} = \int 6u^{-3}du = -3u^{-2} + C = \frac{-3}{u^2} + C = \frac{-3}{(2+sin(t))^2} + C$

$\int \frac{\sec z\tan z}{\sqrt{\sec z}}dz$ The derivative of $sec(z)$ is $sec(z)\,tan(z)$, so set u = sec(z). Then $du = sec(z)\,tan(z)\,dz$ and the integral becomes $\int \frac{du}{\sqrt{u}} = \int u^{-1/2}du = 2u^{1/2} + C = 2\sqrt{u} + C = 2\sqrt{sec\,z} + C$

6. Evaluate the following using integration by parts

$\int t\cos(\pi t)dt$. There derivative of t is 1 (which is simpler) and the integral of cos(t) is sin(t) (not more complicated) set $u = t$ and $dv = cos(\pi t)\,dt$. Then $du\ =\ 1\,dt$ and $v = \frac{1}{\pi}sin(\pi t)$, so integration by parts says that

$\int t\,cos(\pi t)\,dt = \frac{1}{\pi} t\,sin(\pi t) - \int \frac{1}{\pi} sin\,\pi t\,dt = \frac{1}{\pi} t\,sin(\pi t) + \frac{1}{\pi^2} cos\,\pi t\,dt + C$

$\int x^2 \sin x\,dx$. Set $u = x^2$

and $dv = sin(x)\,dx$
Then $du = 2xdx$
and $v = -cos(x)$
so integration by parts says that $\int x^2\,sin(x)\,dx = -x^2\,cos(x) - \int -2x\,cos(x)\,dx = -x^2\,cos(x) + \int 2x\,cos(x)\,dx$
We need to apply integration by parts a second time to find $\int 2x\,cos(x)\,dx$
Set $u = 2x$
and $dv = cos(x)dx$.
Then $du = 2dx$
and $v = sin(x)$ and integration by parts says $\int 2x\,cos(x)\,dx = 2x\,sin(x) - \int 2\,sin(x)\,dx$
$= 2x\,sin(x) + 2\,cos(x) + C$
Putting it all together gives $\int x^2\,sin(x)\,dx = -x^2\,cos(x) + 2x\,sin(x) + 2\,cos(x) + C$

$\int_1^e x^3 \ln x\,dx$ The derivative of $ln\,x$ is $\frac{1}{x}$, which will combine nicely with the antiderivative of x^3. Set $u = ln\,x$ and set $dv = x^3dx$. Then $du = \frac{dx}{x}$ and $v = \frac{x^4}{4}$.
Integration by parts says that $\int x^3\,ln\,x\,dx = \frac{x^4\,ln\,x}{4} - \int \frac{x^4}{4x} dx = \frac{x^4\,ln\,x}{4} - \int \frac{x^3}{4} dx = \frac{x^4\,ln\,x}{4} - \frac{x^4}{16} + C$

$\int \sin^{-1} y\,dy$. If we knew the antiderivative of $sin^{-1}(y)$,

we would be done. Apparently we don't, so set $u = sin^{-1}(y)$. Then $dv = dy$.
Then $du = \frac{1}{1-y^2}$ and $v = y$. Integration by parts gives $\int sin^{-1}(y)\,\mathrm{dy} = \mathrm{y}\,sin^{-1}(y) - \int \frac{y}{\sqrt{1-y^2}} dy$.
To find $\int \frac{y}{\sqrt{1-y^2}} dy$, set $u = 1 - y^2$ so that $du = -2ydy$ or $ydy = -\frac{1}{2} du$. Writing the integral in terms of u gives $\int -\frac{1}{2\sqrt{u}} du = \int -\frac{1}{2} u^{-1/2} = -u^{1/2} + C = -\sqrt{1-y^2} + C$ This tells us that $\int sin^{-1}(y)\,\mathrm{dy} = \mathrm{y}\,sin^{-1}(y) + \sqrt{1-y^2} + C$

$\int e^{-y} \cos y\,dy$. Both e^{-y} and $cos\,y$ are cyclic in their derivatives (and therefore integrals). It takes two derivatives or two integrals for $cos(x)$ to cycle back to a multiple of $cos(x)$, so we'll apply integration by parts twice to get back to an integral similar to the original. We'll then be able to solve for the original integral. Don't forget to add $+C$!
Set $u = e^{-y}$ and $dv = cos\,y\,dy$. Then $du = -e^{-y}$ and $v = sin\,y$, so integration by parts says that $\int e^{-y}\,cos\,y\,dy = e^{-y}\,sin\,y - \int -e^{-y}\,sin\,y\,dy = e^{-y}\,sin\,y + \int e^{-y}\,sin\,y\,dy$. To find $\int e^{-y}\,sin\,y\,dy$ we set $u = e^{-y}$ and $dv = sin\,y\,dy$ so that $\mathrm{du} = -e^{-y}dy$ and $v = -cos(y)$. Applying integration by parts gives $\int e^{-y}\,sin\,y\,dy = -e^{-y}\,cos(y) - \int e^{-y}\,cos(y)$. Thus $\int e^{-y}\,cos\,y\,dy = e^{-y}\,sin\,y - e^{-y}\,cos(y) - \int e^{-y}\,cos(y)$ so $2 \int e^{-y}\,cos(y) = e^{-y}\,sin\,y - e^{-y}\,cos(y)$ so $\int e^{-y}\,cos(y) = \frac{1}{2} e^{-y}\,sin\,y - \frac{1}{2} e^{-y}\,cos(y) + C$